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Level spacing functions and the connection problem of a fifth Painlevé transcendent

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Abstract. In the study of level spacing functions for the eigenvalues of random matrices, Mahoux and Mehta studied the functions $S(t)$, $A(t)$ and $B(t)$ related to certain Fredholm determinants. These functions can be expressed in terms of the fifth (or the third) Painlevé transcendents. The asymptotic behaviour for $t \to \infty$ of these functions and their derivatives with respect to a parameter were derived by them except for an unknown constant. Using Jimbo's method of monodromy preserving deformations, we connect the behaviour of the fifth Painlevé transcendent at $t = 0$ and at $t = \infty$, thus determining the unknown constant.

1. Introduction

In the study of random matrices, the statistics of spacings of the eigenvalues can be expressed in terms of certain Fredholm determinants and their derivatives. These Fredholm determinants in turn are related to the fifth or the third Painlevé transcendents. Our aim is to connect the asymptotic behaviour of these Painlevé transcendents at $t = 0$ and $t = \infty$, thus completing the asymptotic analysis of the Fredholm determinants carried out by Mahoux and Mehta [1].

Nonlinear differential equations appear in many branches of physics, e.g. in statistical mechanics [2], quantum field theory [3] and random matrices [1]. Painlevé considered the general second-order equation $y'' = F\left(t, y, y'\right)$ where the function F is rational in y, y', analytic in t, and the location of any singularities other than simple poles, of the solution $y(t)$ be independent of the initial conditions. By an exhaustive study Painlevé and Gambier [4] found that the solutions of all such equations can be expressed either in terms of the known classical functions (solutions of linear equations and of nonlinear first-order equations) or in terms of solutions of six particular such equations. The solutions of these last mentioned six equations give new transcendental functions which are named as Painlevé-1 (P1), ..., Painlevé-6 (P6).

Many of these physical problems, where one of the Painlevé functions appear, require one to express an asymptotic behaviour of the function in one region in terms of the boundary conditions specified in some other regions. These connection problems have been studied in many ways. One of these techniques makes use of the fact that Painlevé equations also appear as the integrability conditions of certain coupled linear partial differential equations (PDEs) or Lax pairs. We use such 2×2 matrix partial differential equations to study those P5 equations which admit a one-parameter family of solutions at the origin and connect the asymptotic expansion $y(t, z)$ at $t \to \infty, z \to 1$ with the expansion about $t = 0$. A similar connection problem for P5, but with different asymptotic limits, was studied by McCoy and Tang [6] by using WKB techniques where they connect the two-parameter asymptotic expansion $y(t, s, \sigma)$ for a class of P5 at $t \to \infty$ (also for $t \to i\infty$) with the two-parameter expansion about $t \to 0$ (with s and σ as two parameters). The asymptotic behaviour of P5 for $t \to \infty$ and $i\infty$ are the physical regions for the transverse Ising model and Bose gas problems, respectively [6]. The case, we are going to consider in this paper, requires a study of asymptotic behaviour not only at $t \to \infty$ but also at $z \to 1$ (both limits taken simultaneously). It is in this asymptotic limit that P5 appears in the Fredholm determinants related to the spacing distribution of eigenvalues in random matrix theory (RMT). Note here that the parameter z appearing in our case does not remain fixed as $t \to \infty$ (in fact a function of t and z is kept fixed, to be explained later in this paper), contrary to the case considered by McCoy and Tang where the two parameters s and σ are kept fixed at certain values (see [6] for details).

This paper is organized as follows. Subsection 2.1 gives a summary, relating Painlevé equations to the Fredholm determinants appearing in RMT. In subsection 2.2, we briefly review the definition of monodromy data and monodromy preserving deformations. For later use, we also discuss the monodromy data associated with the system of 2×2 PDEs whose coefficient matrices give rise to the fifth Painlevé equation under monodromy preserving condition. The $t \to 0$ limit of P5 has already been worked out by Jimbo [7] and, as we intend to use this result to get rid of unknown constants in our $t \to \infty$ limit of P5, we maintain the same parametrization to describe the monodromy data as that used by Jimbo. Section 3 contains the procedure that we will follow to obtain our result. Section 4 determines the behaviour of PDEs in the $t \to \infty$ limit. Section 5 contains the solutions of these limiting forms of PDEs which finally lead to the $y(t \to \infty)$ solution. Section 6 deals with the special cases of P5 and their application to the random matrix problem. The purpose of this section is to determine the value of a constant α which appears in the derivatives of certain Fredholm determinants (discussed briefly in subsection 2.1). The asymptotic analysis carried out by Mahoux and Mehta [1] for these determinants presents us with a solution with an unknown constant α. By comparison with earlier work, they guess the value to be $\sqrt{\pi}/8$. Section 6 shows that their guess is correct.

2. Prelimnaries

2.1. Relation between RMT and the Painlevé transcendent

It has been shown [1] that the Fredholm determinant

$$F(z, t) = \prod_{i=0}^{\infty} [1 - z\lambda_i(t)] \tag{2.1}$$

of the integral equation

$$\lambda f(x) = \int_{-t}^{t} K(x, y) f(y)\, \mathrm{d}y \tag{2.2}$$

with the kernel K given as

$$K(x, y) = \frac{\sin(x - y)\pi}{(x - y)\pi} \tag{2.3}$$

appears in the spacing distributions of the eigenvalue spectra of RMT.

Consider the function $E_\beta(n,t)$, the probability that a randomly chosen interval of length $2t$, measured in units of the local mean spacing, contain exactly n eigenvalues. For a general value of β, the $E_\beta(n,t)$ are essentially the linear combination of Fredholm determinants [8]. For example, for the unitary ensemble ($\beta = 2$), one has

$$E_2(n,t) = \frac{1}{n!}\left(\frac{-\partial}{\partial z}\right)^n F(z,t)|_{z=1} \tag{2.4}$$

where $F(z,t)$ is given by (2.1).

The relation between the function $E_\beta(n,t)$ and $F(z,t)$ for the orthogonal ensemble ($\beta = 1$) and the symplectic ensemble ($\beta = 4$) is more complicated and contains a linear combination of the latter [1]. In fact, it is convenient to study E_β's for these cases by defining Fredholem determinants F_+ and F_- as follows:

$$F_+ = \prod_{i=0}^{\infty}[1 - z\lambda_{2i}(t)] \tag{2.5}$$

$$F_- = \prod_{i=0}^{\infty}\left[1 - z\lambda_{2i+1}(t)\right] \tag{2.6}$$

where the eigenvalues λ_i are ordered as $0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant 1$ and the corresponding kernels $K_\pm$ appearing in the integral equation (2.2) are

$$K_\pm(x,y) = \tfrac{1}{2}\left[K(x,y) \pm K(-x,y)\right]. \tag{2.7}$$

Mehta [8] further introduces three new functions, namely, $A(z,t)$, $B(z,t)$ and $S(z,t)$ where the first two can be written in terms of $F_\pm$ as

$$A(z,t) = -\frac{1}{2}\frac{\partial}{\partial t}\left[\log F_+(z,t) + \log F_-(z,t)\right] \tag{2.8}$$

$$B(z,t) = -\frac{1}{2}\frac{\partial}{\partial t}\left[\log F_+(z,t) - \log F_-(z,t)\right] \tag{2.9}$$

and the third, namely S, is completely determined by the nonlinear differential equation

$$\frac{\mathrm{i}\,\mathrm{d}S}{\mathrm{d}t} + \pi S = \frac{z}{2\pi t}S^\star\left(S^2 - (S^\star)^2\right). \tag{2.10}$$

and the initial condition $S(z,0) = 1$. Any one of these functions S, A and B completely determine the two other and each of them satisfies a nonlinear differential equation. A detailed information about these functions can be found in [1]. It is the relation of these functions S, A and B with the fifth or third Painlevé transcendent (as shown in [1]) which formally relates the fluctuation measures of RMT with the latter. For example, S can be expressed as follows (for simplicity of notation, we express S in terms of

$$\tau = \pi t \qquad \zeta = 2z/\pi \tag{2.11}$$

$$S = \frac{(R + \mathrm{i}I)}{\sqrt{\zeta}} \tag{2.12}$$

with R and I real. If one writes the following equations:

$$(R+\sqrt{\tau}) = y_r(R-\sqrt{\tau}) \tag{2.13}$$

$$(I+\sqrt{\tau}) = y_i(I-\sqrt{\tau}) \tag{2.14}$$

then both y_r and y_i satisfy the fifth Painlevé equation which is

$$\frac{d^2y}{d\tau^2} = \left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{d\tau}\right)^2 - \frac{1}{\tau}\frac{dy}{d\tau} + \frac{(y-1)^2}{\tau^2}\left(\alpha y+\frac{\beta}{y}\right) + \gamma\frac{y}{\tau} + \delta\frac{y(y+1)}{(y-1)} \tag{2.15}$$

with constants $\alpha = -\beta = \frac{1}{32}$, $\gamma = 0$ and $\delta = -2$. Similarly one can relate A and B with P5 or P3 with different values the of constants α, β, γ and δ [1].

To gain an insight in the asymptotic behaviour of spacing distributions of RMT, Mahoux and Mehta studied the asymptotic expansion of $S(\zeta,\tau)$ around $\zeta = 2/\pi$ and for large positive values of τ, which can be written as follows, (recall here that $\zeta = 2/\pi$ implies $z = 1$, a value at which we need to evaluate the derivative of F, to calculate E_β; see equation (2.4)):

$$S(\zeta,\tau) = 2\sqrt{\frac{\tau}{\zeta}}\sum_{n=0}^{\infty}\left[\alpha_1\left(\zeta-\frac{2}{\pi}\right)\frac{e^{2\tau}}{\sqrt{\tau}}\right]^n S_n(\tau) \tag{2.16}$$

with an undetermined constant α_1; here $S_n(\tau) = R_n(\tau)+iI_n(\tau)$ with both R_n and I_n having asymptotic expansions in $1/\tau$ [1].

The need to determine the value of the constant α_1 persuades us to undertake the present study where we first study the behaviour of P5 under similar limiting conditions, namely

$$\tau\to\infty \qquad \zeta\to 2/\pi \qquad \left(\zeta-\frac{2}{\pi}\right)e^{2\tau}/\sqrt{\tau} = \text{constant} \tag{2.17}$$

and then use the following equality to calculate α_1:

$$\begin{aligned} y_r(\tau,\zeta)&\left[R_0(\tau)-\tfrac{1}{2}+\sum_{n=1}^{\infty}\left[\alpha_1\left(\zeta-\frac{2}{\pi}\right)\frac{e^{2\tau}}{\sqrt{\tau}}\right]^n R_n(\tau)\right] \\ &= \left[R_0(\tau)+\tfrac{1}{2}+\sum_{n=1}^{\infty}\left[\alpha_1\left(\zeta-\frac{2}{\pi}\right)\frac{e^{2\tau}}{\sqrt{\tau}}\right]^n R_n(\tau)\right] \end{aligned} \tag{2.18}$$

where equation (2.18) follows from (2.12), (2.13) and (2.16).

2.2. *Generalities on Painlevé transcendent*

Earlier studies [5] on Painlevé equations have shown that for each Painlevé trascendent, a set of 2×2 linear partial differential equations (PDEs) can be defined where the former appear in the coefficient matrices of the later. This allows us to study the asymptotic behaviour of these trancendents by analysing the related PDEs in the corresponding asymptotic limit. We will discuss here the case for fifth Painlevé transcendent only [7].

Consider the following 2×2 system of linear partial differential equations:

$$\frac{\partial \psi(x,\tau)}{\partial x} = \left[\frac{A_0(\tau)}{x} + \frac{A_1(\tau)}{x-\tau} + \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right]\psi(x,\tau) \tag{2.19}$$

$$\frac{\partial \psi(x,\tau)}{\partial \tau} = -\frac{A_1(\tau)}{x-\tau}\psi(x,\tau) \tag{2.20}$$

where A_0 and A_1 are 2×2 matrices

$$A_0(\tau) = \begin{bmatrix} \frac{\theta_0}{2} + v & -u\left(\theta_0 + v\right) \\ u^{-1}v & -\left(\frac{\theta_0}{2} + v\right) \end{bmatrix} \tag{2.21}$$

$$A_1(\tau) = \begin{bmatrix} -\left(\frac{\theta_\infty}{2} + \frac{\theta_0}{2} + v\right) & -uy\left(\frac{\theta_\infty}{2} + \frac{\theta_0}{2} + \frac{\theta_1}{2} + v\right) \\ u^{-1}y^{-1}\left(\frac{\theta_\infty}{2} - \frac{\theta_1}{2} + \frac{\theta_0}{2} + v\right) & \left(\frac{\theta_\infty}{2} + \frac{\theta_0}{2} + v\right) \end{bmatrix} \tag{2.22}$$

$$\mathrm{Diag}\left[A_0(\tau) + A_1(\tau)\right] = -\frac{1}{2}\begin{pmatrix} \theta_\infty & \\ & -\theta_\infty \end{pmatrix} \tag{2.23}$$

where the eigenvalues of A_0 and A_1 are $\pm\frac{1}{2}\theta_0$, $\pm\frac{1}{2}\theta_1$; and v, u, y are functions of τ. The $\psi(x,\tau)$ in the above is a 2×2 matrix. The integrability condition of equations (2.19) and (2.20) imply that

$$y \equiv y(\tau) = \frac{\left[(A_1(\tau))_{12}\right]\left[(A_0(\tau))_{11} + \frac{\theta_0}{2}\right]}{\left[(A_0(\tau))_{12}\right]\left[(A_1(\tau))_{11} + \frac{\theta_1}{2}\right]} \tag{2.24}$$

satisfies the nonlinear ordinary differential equation (2.15) with constants $\alpha, \beta, \gamma, \delta$ given by

$$\begin{aligned} \alpha &= \tfrac{1}{8}\left(\theta_0 - \theta_1 + \theta_\infty\right)^2 & \gamma &= 1 - \theta_0 - \theta_1 \\ \beta_1 &= -\tfrac{1}{8}\left(\theta_0 - \theta_1 - \theta_\infty\right)^2 & \delta &= -\tfrac{1}{2}. \end{aligned} \tag{2.25}$$

Note here that equations (2.13)–(2.18), shown in subsection 2.1, are still valid (derived for $y(\tau)$, satisfying equation (2.15) with $\delta = -2$) for $y(\tau)$ if we replace τ by $\tau/2$ in each of these relations. Equations (2.19) and (2.20) have regular singularities at $x = 0$ and $x = \tau$ and an irregular singularity at $x = \infty$. We denote them as a_0, a_1 and a_∞. In general, $\psi(x,\tau)$ is a multivalued function of the complex variable x. Let γ_j be a closed path encircling once the only singularity at a_j, in the positive sense for γ_0 and γ_1, and in the negative sense for γ_∞. The analytical continuation of $\psi(x,\tau)$ around the singular point a_j following the path γ_j gives

$$\psi\left(\gamma_j x\right) = \psi(x)M_j \tag{2.26}$$

where x becomes $\gamma_j x$ following the path γ_j. M_j is the so-called monodromy matrix at a_j. With a convenient choice of the paths γ_0, γ_1 and γ_∞ one has

$$M_\infty M_1 M_0 = 1. \tag{2.27}$$

Besides, due to irregular singularity at $x = \infty$, we have the Stokes phenomenon.

Let S_j be the sector $\left\{x \in C \,\middle|\, -\frac{\pi}{2}(2j-1) < \arg x < \frac{\pi}{2}(-2j+5)\right\}$. We denote by $\psi_j(x,\tau)$ the solution of (2.19), (2.20) uniquely defined by its asymptotic behaviour in sector S_j:

$$\psi_j(x,\tau) \sim \hat{\psi} = \left(1 + \mathrm{O}\left(x^{-1}\right)\right) x^{-\frac{1}{2}\begin{pmatrix}\theta_\infty & \\ & -\theta_\infty\end{pmatrix}} \exp\begin{pmatrix} x & \\ & 0\end{pmatrix}. \qquad (2.28)$$

The solutions $\psi_j(x,\tau)$ and $\psi_{j+1}(x,\tau)$ are connected by the so-called Stokes multipliers. In particular we have

$$\psi_2(x,\tau) = \psi_1(x,\tau)\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix} \qquad (2.29)$$

and

$$\psi_3(x,\tau) = \psi_2(x,\tau)\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}. \qquad (2.30)$$

In what follows we will be interested in the solution $\psi_1(x,\tau)$ which we will indicate as $\psi(x,\tau)$.

The local behaviour of $\psi(x,\tau)$ near the regular singularities $x=0$ and $x=\tau$ has the form

$$\psi_1(x,\tau) = G_0(\tau)[1+\mathrm{O}(x)]x^{\frac{1}{2}\begin{pmatrix}\theta_0 & \\ & -\theta_0\end{pmatrix}} C_0 \qquad (2.31)$$

$$= G_1(\tau)[1+\mathrm{O}(x-\tau)](x-\tau)^{\frac{1}{2}\begin{pmatrix}\theta_1 & \\ & -\theta_1\end{pmatrix}} C_1 \qquad (2.32)$$

where G_0 and G_1 diagonalize A_0 and A_1

$$A_0 = G_0\begin{pmatrix}\frac{\theta_0}{2} & \\ & \frac{-\theta_0}{2}\end{pmatrix} G_0^{-1} = G_0 T_0 G_0^{-1} \qquad (2.33)$$

$$A_1 = G_1\begin{pmatrix}\frac{\theta_1}{2} & \\ & \frac{-\theta_1}{2}\end{pmatrix} G_1^{-1} = G_1 T_1 G_1^{-1} \qquad (2.34)$$

and C_0 and C_1 are the so-called connection matrices. The monodromy matrices M_0 and M_1 are given as

$$M_0 = C_0^{-1} \mathrm{e}^{2\pi \mathrm{i} T_0} C_0 \qquad (2.35)$$

$$M_1 = C_1^{-1} \mathrm{e}^{2\pi \mathrm{i} T_1} C_1. \qquad (2.36)$$

From equations (2.26)–(2.30), the monodromy matrix M_∞ is given by

$$M_\infty^{-1} = \begin{pmatrix}1 & b\\ a & 1+ab\end{pmatrix} \mathrm{e}^{-\mathrm{i}\pi\begin{pmatrix}\theta_\infty & \\ & -\theta_\infty\end{pmatrix}}. \qquad (2.37)$$

It has been shown by Jimbo [7] that if we set

$$\mathrm{e}^{\pi \mathrm{i}\theta_\infty} ab + 2\cos\pi\theta_\infty = 2\cos\pi\sigma \qquad 0 \leqslant \mathrm{Re}\,\sigma \leqslant 1 \qquad (2.38)$$

then the connection matrices can be parametrized as follows ($\sigma \neq 0$)

$$D^{(1)}C_1 D = \begin{pmatrix} \sin\frac{\pi}{2}(\theta_1+\theta_0+\sigma) & -s\ \sin\frac{\pi}{2}(\theta_1+\theta_0-\sigma) \\ \times\sin\frac{\pi}{2}(\theta_1-\theta_0+\sigma) & \times\sin\frac{\pi}{2}(\theta_1-\theta_0-\sigma) \\ -s^{-1} & 1 \end{pmatrix} \times \begin{pmatrix} e^{-\pi i\sigma/2} & \sin\frac{\pi}{2}(\theta_\infty+\sigma) \\ e^{\pi i\sigma/2} & \sin\frac{\pi}{2}(\theta_\infty-\sigma) \end{pmatrix} \tag{2.39}$$

$$D^{(0)}C_0 D = \begin{pmatrix} \sin\frac{\pi}{2}(\theta_1+\theta_0+\sigma) & -s\ e^{-\pi i\sigma}\sin\frac{\pi}{2}(\theta_1+\theta_0-\sigma) \\ -s^{-1}e^{\pi i\sigma}(\theta_1-\theta_0+\sigma) & \sin\frac{\pi}{2}(\theta_1-\theta_0-\sigma) \end{pmatrix} \times \begin{pmatrix} e^{-\pi i\sigma/2} & \sin\frac{\pi}{2}(\theta_\infty+\sigma) \\ e^{\pi i\sigma/2} & \sin\frac{\pi}{2}(\theta_\infty-\sigma) \end{pmatrix} \tag{2.40}$$

and for later use we set

$$\hat{s} = s\frac{\Gamma\left\{\frac{1}{2}(\theta_1+\theta_0+\sigma)+1\right\}\Gamma\left\{\frac{1}{2}(\theta_1-\theta_0+\sigma)+1\right\}\Gamma\left\{\frac{1}{2}(\theta_\infty+\sigma)+1\right\}\Gamma\left\{(1-\sigma)^2\right\}}{\Gamma\left\{\frac{1}{2}(\theta_1+\theta_0-\sigma)+1\right\}\Gamma\left\{\frac{1}{2}(\theta_1-\theta_0-\sigma)+1\right\}\Gamma\left\{\frac{1}{2}(\theta_\infty-\sigma)+1\right\}\Gamma\{(1+\sigma)^2\}}. \tag{2.41}$$

Here $D^{(1)}, D^{(0)}$ and D are invertible diagonal matrices†. The connection matrices (given by (2.31) and (2.32)) related to these singularities, along with the θ's form the monodromy data which is invariant with respect to τ; see [5] for a detailed description. The knowledge of this monodromy data and its invariant nature, along with the singularity data, has been used to study the behaviour of y in $\tau \to 0$ by Jimbo [7]; we will use a similar technique to study the behaviour for $\tau \to \infty$.

3. Procedure

The Painlevé transcendent y appearing in the random matrix theory depends on an extra parameter ζ through the initial conditions at $\tau = 0$. To emphasize this we will sometimes indicate this dependence explicitly.

Our aim is to determine $y(\tau, \zeta)$, equation (2.24), which is a solution of (2.15), when

$$\tau \to \infty \qquad \zeta \to 2/\pi \qquad \left(\zeta - \frac{2}{\pi}\right)e^{\tau}/\sqrt{\tau} = \text{constant} \tag{3.1}$$

by using the information available for $y(\tau \to 0, \zeta)$. For this purpose (as is obvious from (2.24)), we need to determine two unknowns $A_0(\tau)$ and $A_1(\tau)$ under the same limit. This further requires two equations in terms of A_0 and A_1. The prior information we have about A_0 and A_1 is that they appear on the right-hand sides of (2.19) and (2.20) with properties given by (2.33) and (2.34). The need to determine A's in limit (3.1) makes our job easier as then it is possible to define two functions of ψ which are separable in x and τ. This, along with the PDEs for ψ (equations (2.19), (2.20)), leads us to ordinary differential equations of these functions whose right-hand sides contain a linear relation of A_0 and

† As the solution y of (2.15) depends on a fixed set of constants $(\alpha, \beta, \gamma, \delta)$ which are related to θ_0, θ_1 and θ_∞ (equation (2.25)), so the θ's are independent of τ; also each Painlevé transcendent is related to a particular set of 2×2 PDEs (i.e. with a particular singularity structure).

A_1 (or only one of them) and are comparatively easier to solve due to their one-variable dependence. Using knowledge gained about the local solutions of these functions (obtained by their relationship with $\psi(x,\tau,\zeta)$ while its local solutions are given by (2.28)–(2.32)) and their differential equations, we obtain two linear relations of A_0 and A_1.

This can be achieved by the following steps:

(i) We first define a function $\phi(x,\tau,\zeta) = \psi(x,\tau,\zeta)G_1x^{-A_1}$ the differential equation of which, with respect to x, has a limit under condition (3.1). The limiting function $\phi(x,\tau)$ is separable in the variables x and τ.

(ii) Step (i) shows that $\phi(x,\tau)$ has only one regular singularity $x = 0$ and one irregular singularity. As the monodromy data remain unchanged under the variation in τ, we use monodromy data (obtained by Jimbo by solving (2.19), (2.20) as $\tau \to 0$) to find the local solutions of $\phi(x,\tau)$ around the singularities in x, which further leads to the determination of $\phi(x,\tau)$ for all x (as for given coefficient matrices A_j and monodromy data, there exist only unique local solution in x).

(iii) But knowledge of $\phi(x,\tau)$ is not sufficient as it does not give us both $A_0(\tau \to \infty, \zeta \to 2/\pi)$ and $A_1(\tau \to \infty, \zeta \to 2/\pi)$ separately. In fact, we need to define another function $\tilde{\psi}(x,\tau)$:

$$\tilde{\psi}(x,\tau,\zeta) = \left(\frac{x}{\tau} + \tau\right)^{\Lambda} e^{-G(x/\tau+\tau)}\psi\left(\frac{x}{\tau} + \tau, \tau, \zeta\right) \tag{3.2}$$

where Λ is obtained from step (ii). Then, by repetition of steps (i) and (ii) with $\tilde{\psi}(x,\tau)$ and its differential equation, we obtain separate values of $A_0(\tau \to \infty)$ and $A_1(\tau \to \infty)$.

4. Determination of ordinary differential equations in the $\tau \to \infty, \zeta \to \frac{2}{\pi}$ limit

Let us define a function $\phi(x,\tau,\zeta)$ where

$$\phi(x,\tau,\zeta) = \psi(x,\tau,\zeta)G_1^{-1}(\tau)x^{-A_1(\tau)} \tag{4.1}$$

$$= \psi(x,\tau,\zeta)x^{-T_1}G_1^{-1} \tag{4.2}$$

where $A_1(\tau)$ and T_1 are defined by (2.22) and (2.34). Recall that T_1 is a constant diagonal matrix, composed of eigenvalues of A_1. For simplicity of presentation, we will not be writing the functional dependence on ζ everywhere and assume it to be understood.

The partial differentiation of (4.1), with respect to x, leads us to the following equation:

$$\frac{\partial\psi(x,\tau,\zeta)}{\partial x} = \frac{\partial\phi(x,\tau,\zeta)}{\partial x}x^{A_1}G_1 + \phi\frac{A_1}{x}x^{A_1}G_1 \tag{4.3}$$

as A_1 and T_1 are independent of x.

Now as our aim is to have, firstly, a differential equation only in one variable, namely, x, and, secondly a linear relation consisting only of A_0 and A_1, we make the following assumptions.

Under the limit (3.1), (i) $[\phi(x,\tau), A_1(\tau)] \to 0$, and (ii) $A_1(\tau)/\tau \to 0$, where (i) is required for having a linear relation in A_0 and A_1 and (ii) is needed for ψ to be separable in the variables. Using these assumptions, it can be shown (by using (2.19)) that

$$\lim \frac{\partial\phi}{\partial x} = \left[\frac{A_0(\tau) - A_1(\tau)}{x} + \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right]\phi. \tag{4.4}$$

Under the limit (3.1) and by using the above assumptions, equation (2.20) can be used to show (appendix 1) that $\psi(x,\tau)$ and hence $\phi(x,\tau)$ is separable in x and τ, that is, we can write $\phi(x,\tau)$ as follows:

$$\phi(x,\tau) = \varphi(x)G_2(\tau). \tag{4.5}$$

For later purposes, it is convenient to introduce a function $G_3(\tau)$ such that $G_3\varphi(x) = \varphi(x)G_3$ and $G_2G_1 = G_3$. Substitution of (4.5) in (4.4) shows that $\varphi(x)$ also satisfies equation (4.4). But as $\varphi(x)$ depends only on x and not on τ, the differential equation governing its behaviour would not contain any τ-dependence. As the eqs.(4.4) and (4.5) are the results of assumptions (i,ii), the latter further leads us to the following relation (under limit (3.1)).

$$\lim A_0(\tau) - A_1(\tau) = \lambda \tag{4.6}$$

where λ does not depend on τ. Equation (4.6) therefore give us one relation between A_0 and A_1, but we need one more to calculate A_0 and A_1 separately.

Let us define another function

$$\tilde{\psi}(x,\tau,\zeta) = \left(\frac{x}{\tau}+\tau\right)^{\Lambda} \mathrm{e}^{-G(x/\tau+\tau)}\psi\left(\frac{x}{\tau}+\tau,\tau,\zeta\right) \tag{4.7}$$

where

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \Lambda \equiv \Lambda(\tau) = -\mathrm{e}^{-G\tau}\lambda\mathrm{e}^{G\tau}. \tag{4.8}$$

The whole idea behind choosing such a complicated-looking function lies in the following: (i) under the limit (3.1), $\tilde{\psi}$ is separable in x and τ (see appendix 2); also by using (2.20) one can show that under condition (3.1)

$$\frac{\partial\tilde{\psi}}{\partial\tau} \approx \Lambda_1 \log\tau\,\tilde{\psi} \tag{4.9}$$

where Λ_1 is defined in appendix 2. Thus under the same limit, $\tilde{\psi}$ can be expressed as follows:

$$\tilde{\psi}(x,\tau,\zeta) = \tilde{\phi}(x,\zeta)\, T\exp\left[\int \Lambda_1 \log\tau\,\mathrm{d}\tau\right] \tag{4.10}$$

which implies that

$$\tilde{\phi}(x,\zeta) = \left(\frac{x}{\tau}+\tau\right)^{\Lambda} \mathrm{e}^{-G(\frac{x}{\tau}+\tau)}\psi\left(\frac{x}{\tau}+\tau,\tau,\zeta\right)\, T\exp\left[-\int \Lambda_1 \log\tau\,\mathrm{d}\tau\right] \tag{4.11}$$

where 'T' before 'exp' implies the τ-ordering of integration.
(ii) This definition of $\tilde{\phi}$ leads us to a differential equation which contains only a function of A_1 thus giving the second required relation to calculate A_0 and A_1 separately. By using (4.11), (4.6), (4.8) and (2.19), it is easy to verify that under condition (3.1)

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}x} = \frac{\tau^{\Lambda}\mathrm{e}^{-G\tau}A_1(\tau)\mathrm{e}^{G\tau}\tau^{-\Lambda}}{x}\tilde{\phi} = \frac{A_1^{\infty}}{x}\tilde{\phi} \tag{4.12}$$

where

$$A_1^{\infty} = \lim\, \tau^{\Lambda}\mathrm{e}^{-G\tau}A_1(\tau)\mathrm{e}^{G\tau}\tau^{-\Lambda}. \tag{4.13}$$

Thus the behaviour of $A_1(\tau)$ and $A_0(\tau)$ under condition (3.1) can be determined by knowing A_1^{∞} and $\Lambda(\tau)$. In the next section, we solve equations (4.4) and (4.12) to give us A_1^{∞} and $\Lambda(\tau \to \infty)$.

5. Determination of λ and A_1^∞ by using the monodromy invariance property

5.1. Determination of λ

As follows from (4.6), we need to determine λ to know about $A_0(\tau)$ and $A_1(\tau)$. As follows from (4.4), (4.5), λ appears in the following differential equation satisfied by $\varphi(x)$:

$$\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} = \left[\frac{\lambda}{x} + \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right]\varphi(x) \tag{5.1}$$

and hence can be calculated by knowing $\varphi(x)$. The latter can be written by explioting the information about the local behaviour of $\psi(x)$ (as given by (2.28)–(2.30)) and the following relation (equations (4.1), (4.5)):

$$\begin{aligned}\varphi(x)G_3(\tau) &= G_3(\tau)\varphi(x)\\ &= \phi(x,\tau,\zeta)G_1(\tau)\\ &= \psi(x,\tau,\zeta)x^{-T_1}.\end{aligned} \tag{5.2}$$

As the Stokes phenomena and asymptotic behaviour of ψ (given by (2.28)–(2.30)) are fixed for a given set of monodromy data (θ's and connection matrices), relation (5.2) between φ and ψ requires that $\varphi(x)$ should have the following asymptotic behaviour and Stokes phenomena:

$$\begin{aligned}\varphi_j(x) &\sim \hat{\varphi} = G_3^{-1}\psi x^{-T_1}\\ &= G_3^{-1}\left(1+0\left(x^{-1}\right)\right)x^{-\frac{1}{2}\begin{pmatrix}\theta_\infty+\theta_1 & \\ & -\theta_\infty-\theta_1\end{pmatrix}}\mathrm{e}^{\begin{pmatrix} x & \\ & 0\end{pmatrix}} \qquad (x\to\infty \text{ in } S_j)\\ \varphi_2(x) &= G_3^{-1}\psi_2 x^{-T_1}.\end{aligned} \tag{5.3}$$

Now, by using the commutability of ϕ with A_1 in the limit (3.1) and equation (4.1), one can show that $\psi x^{-T_1} = x^{-A_1}\psi$. This gives us the following equality:

$$\begin{aligned}\varphi_2(x) &= G_3^{-1}x^{-A_1}\psi_2\\ &= G_3^{-1}x^{-A_1}\psi_1\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}\\ &= G_3^{-1}\psi_1 x^{-T_1}\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}\\ &= \varphi_1\begin{pmatrix}1 & 0\\ a & 1\end{pmatrix}.\end{aligned} \tag{5.4}$$

Similarly one has

$$\varphi_3(x) = \varphi_2(x)\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}. \tag{5.5}$$

All these requirements are fulfilled if one considers the following form of $\varphi(x)$:

$$\varphi_{\frac{1}{2}}(x) = R^{-1} W_{\frac{1}{2}}(x) R$$

$$W_{\frac{1}{2}}(x) = \begin{bmatrix} \mathrm{e}^{\pm\pi\mathrm{i}(1-\theta_\infty-\theta_1)/2} & -\frac{1}{2}(\sigma-\theta_\infty-\theta_1) \\ \times W_{(1-\theta_\infty-\theta_1)/2,\sigma/2}\left(\mathrm{e}^{\mp\pi\mathrm{i}}\right) & \times W_{(-1+\theta_\infty+\theta_1)/2,\sigma/2}(x) \\ -\frac{1}{2}(\sigma+\theta_\infty+\theta_1)\,\mathrm{e}^{\pm\pi\mathrm{i}(1-\theta_\infty-\theta_1)/2} & \\ \times W_{(-1+\theta_\infty+\theta_1)/2,\sigma/2}\left(\mathrm{e}^{\pm\pi\mathrm{i}}x\right) & W_{(1+\theta_\infty+\theta_1)/2,\sigma/2}(x) \end{bmatrix}$$
$$\times\, x^{-1/2}\mathrm{e}^{x/2} \tag{5.6}$$

where $W_{jk}(x)$ is the Whittaker function [9] and the matrix R is introduced to maintain the same Stokes multipliers for φ as those of ψ (see appendix 3). Here R is chosen to be

$$R = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$$

with $r = a_1/a$, where a_1 and b_1 are the Stokes multiplier of the function W [7], namely

$$a_1 = \frac{2\pi\mathrm{i}}{\Gamma\left\{\frac{\sigma+\theta_\infty+\theta_1}{2}\right\}\Gamma\left\{1-\frac{\sigma-\theta_\infty-\theta_1}{2}\right\}} \tag{5.7}$$

$$b_1 = \frac{2\pi\mathrm{i}\mathrm{e}^{-\pi\mathrm{i}(\theta_\infty+\theta_1)}}{\Gamma\left\{1-\frac{\sigma+\theta_\infty+\theta_1}{2}\right\}\Gamma\left\{\frac{\sigma-\theta_\infty-\theta_1}{2}\right\}}. \tag{5.8}$$

The behaviour near the regular singularity $x = 0$ is given by

$$\varphi_1(x) = R^{-1} G_\lambda (1 + \mathrm{O}(x)) x^{\frac{1}{2}\begin{pmatrix} \sigma & \\ & -\sigma \end{pmatrix}} C_\lambda R \tag{5.9}$$

where

$$G_\lambda = \begin{pmatrix} \frac{1}{2}(\sigma-\theta_\infty-\theta_1) & -1 \\ \frac{1}{2}(\sigma+\theta_\infty+\theta_1) & 1 \end{pmatrix}$$

$$C_\lambda = \begin{bmatrix} \dfrac{-\Gamma\{-\sigma\})\mathrm{e}^{-\pi\mathrm{i}(\sigma+\theta_\infty+\theta_1)/2}}{\Gamma\left\{1-\frac{\sigma-\theta_\infty-\theta_1}{2}\right\}} & \dfrac{-\Gamma\{-\sigma\}}{\Gamma\left\{1-\frac{\sigma+\theta_\infty+\theta_1}{2}\right\}} \\ \dfrac{-\Gamma\{\sigma\}\mathrm{e}^{\pi\mathrm{i}(\sigma-\theta_\infty-\theta_1)/2}}{\Gamma\left\{\frac{\sigma+\theta_\infty+\theta_1}{2}\right\}} & \dfrac{\Gamma\{\sigma\}}{\Gamma\left\{\frac{\sigma-\theta_\infty-\theta_1}{2}\right\}} \end{bmatrix}. \tag{5.10}$$

Now by differentiating $\varphi_j(x)$ with respect to x, it can be shown (appendix 4) that it satisfies the following differential equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = R^{-1}\left[\frac{1}{2}\begin{pmatrix} -\theta_\infty-\theta_1 & \sigma-\theta_\infty-\theta_1 \\ \sigma+\theta_\infty+\theta_1 & \theta_\infty+\theta_1 \end{pmatrix}\frac{1}{x} + \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right] R\varphi(x). \tag{5.11}$$

Comparison of (5.1) and (5.11) gives

$$R\lambda R^{-1} = \frac{1}{2}\begin{pmatrix} -\theta_\infty-\theta_1 & \sigma-\theta_\infty-\theta_1 \\ \sigma+\theta_\infty+\theta_1 & \theta_\infty+\theta_1 \end{pmatrix}. \tag{5.12}$$

5.2. Determination of A_l^∞

From relation (4.7), it is easy to see (appendix 5) that the monodromy data of $\tilde{\psi}(x,\tau,\zeta)$ around the singularity $x = 0$ are the same as that for $\psi(x,\tau,\zeta)$ around the singularity $x=\tau$. We already know the local behaviour of ψ, around $x=\tau$ (equation (2.32)) and the related connection matrix C_1. Thus the local behaviour of $\tilde{\psi}(x,\tau)$, in the $\tau\to\infty$ limit, can be written as follows:

$$\begin{aligned}\lim \tilde{\psi}(x,\tau,\zeta)\, T\exp\left[-\int \Lambda_1 \log\tau\, \mathrm{d}\tau\right] &= \tilde{\phi}(x,\zeta)\\ &= G_1^\infty(1+\mathrm{O}(x))x^{\frac{1}{2}\begin{pmatrix}\theta_1 & \\ & -\theta_1\end{pmatrix}}C_1 \qquad (x\to 0)\\ &= G_1^\infty\left(1+\mathrm{O}\left(x^{-1}\right)\right)x^{\frac{1}{2}\begin{pmatrix}\theta_1 & \\ & -\theta_1\end{pmatrix}}C_1 \qquad (x\to\infty).\end{aligned} \tag{5.13}$$

Let us consider the following equation:

$$\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}x} = \frac{1}{x}\frac{1}{2}\begin{pmatrix}\theta_1 & \\ & -\theta_1\end{pmatrix}\tilde{\varphi} \tag{5.14}$$

with local and asymptotic behaviours

$$\begin{aligned}\tilde{\varphi}(x,\zeta) &= (1+\mathrm{O}(x))x^{\frac{1}{2}\begin{pmatrix}\theta_1 & \\ & -\theta_1\end{pmatrix}} \qquad (x\to 0)\\ &= \left(1+\mathrm{O}\left(x^{-1}\right)\right)x^{\frac{1}{2}\begin{pmatrix}\theta_1 & \\ & -\theta_1\end{pmatrix}} \qquad (x\to\infty).\end{aligned} \tag{5.15}$$

Comparison of equation (5.13) and (5.15) shows that

$$\lim \tau^{\Lambda}\mathrm{e}^{-G\tau}\psi(x/\tau+\tau,\tau)\, T\exp\left[-\int \Lambda_1\log\tau\,\mathrm{d}\tau\right] = G_1^\infty\tilde{\varphi}(x)C_1. \tag{5.16}$$

If for further simplification, we redefine $\tilde{\psi}$ as follows:

$$\tilde{\psi}(x,\tau,\zeta) = \left(\frac{x}{\tau}+\tau\right)^{\Lambda}\mathrm{e}^{-G(x/\tau+\tau)}R\psi\left(\frac{x}{\tau}+\tau,\tau,\zeta\right)R^{-1} \tag{5.17}$$

then the expressions of appendices 2 and 3, dealing with the behaviour of $\tilde{\psi}$ (defined by (4.7)), still remain valid for $\tilde{\psi}$ defined by (5.17), except that the definitions of Λ and A_1^∞ are now given as follows:

$$A_1^\infty = \lim \tau^{\Lambda}\mathrm{e}^{-G\tau}RA_1(\tau)R^{-1}\mathrm{e}^{G\tau}\tau^{-\Lambda} \tag{5.18}$$

where Λ is defined as

$$\Lambda = -\mathrm{e}^{-G\tau}R\lambda R^{-1}\mathrm{e}^{G\tau}. \tag{5.19}$$

The connection matrix C_1 around the singularity $x = \tau$ is given by (2.39). As was shown by Jimbo [7] for the $\tau \to 0$ case (around equation (3.10) of his paper), for C_1 to have a form like (2.39), G_1 and C_1 should have the form

$$G_1^\infty = \frac{1}{\sigma}\begin{pmatrix} \frac{1}{2}(\sigma-\theta_\infty) & \frac{2\beta_1}{\hat{s}(\sigma+\theta_\infty)} \\ \frac{1}{2}(\sigma+\theta_\infty)\gamma_1 & \frac{-2\beta_1\gamma_1}{\hat{s}(\sigma+\theta_\infty)} \end{pmatrix}\begin{pmatrix} 1 & -\frac{1}{4}(\theta_0+\theta_1-\sigma)(\theta_0-\theta_1+\sigma) \\ 1 & -\frac{1}{4}(\theta_0+\theta_1+\sigma)(\theta_0-\theta_1-\sigma) \end{pmatrix}$$
$$= \frac{1}{\sigma}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.20a}$$

and

$$C_1 \equiv \begin{bmatrix} \frac{-\Gamma(-\theta_1)\Gamma(1-\sigma)}{\Gamma\left(\frac{1}{2}(\theta_0-\theta_1-\sigma)\right)\Gamma\left(-\frac{1}{2}(\theta_1+\theta_0+\sigma)\right)} & \frac{-\hat{s}\Gamma(-\theta_1)\Gamma(1+\sigma)(\sigma+\theta_\infty)/2\beta_1}{\Gamma\left(\frac{1}{2}(\theta_0-\theta_1+\sigma)\right)\Gamma\left(-\frac{1}{2}(\theta_0+\theta_1-\sigma)\right)} \\ \frac{-e^{-\pi i\theta_1}\Gamma(\theta_1)\Gamma(1-\sigma)}{\Gamma\left(1+\frac{1}{2}(\theta_1+\theta_0-\sigma)\right)\Gamma\left(1-\frac{1}{2}(\theta_0-\theta_1+\sigma)\right)} & \frac{-e^{-\pi i\theta_1}\Gamma(\theta_1)\Gamma(1-\sigma)\hat{s}(\sigma+\theta_\infty)/2\beta_1}{\Gamma\left(1+\frac{1}{2}(\theta_1+\theta_0+\sigma)\right)\Gamma\left(1-\frac{1}{2}(\theta_0-\theta_1-\sigma)\right)} \end{bmatrix}$$
$$\times\begin{bmatrix} \frac{-\Gamma(-\sigma)}{\Gamma\left(1-\frac{\sigma-\theta_\infty}{2}\right)}e^{-\pi i\left(\frac{\sigma+\theta_\infty}{2}\right)} & \frac{-\Gamma(-\sigma)\gamma_1}{\Gamma\left(1-\frac{\sigma+\theta_\infty}{2}\right)} \\ \frac{-\Gamma(\sigma)}{\Gamma\left(\frac{\sigma+\theta_\infty}{2}\right)}e^{\pi i\left(\frac{\sigma-\theta_\infty}{2}\right)} & \frac{\Gamma(\sigma)\gamma_1}{\Gamma\left(\frac{\sigma-\theta_\infty}{2}\right)} \end{bmatrix}. \tag{5.20b}$$

where

$$\gamma_1 = \frac{-2(1+(\sigma+\theta_0)/2)}{(1-\sigma-\theta_0)}\sin\pi\left(\frac{\sigma+\theta_\infty}{2}\right)\Gamma(\sigma) \qquad \beta_1 = \frac{\theta_1}{2}(1-\sigma-\theta_0)\Gamma(\sigma).$$

Using relation (5.16) in (4.12) and comparing with (5.14) gives the following result:

$$A_1^\infty = G_1^\infty\begin{pmatrix} \theta_1/2 & \\ & -\theta_1/2 \end{pmatrix}\left(G_1^\infty\right)^{-1}. \tag{5.21}$$

Using (5.20), (5.18) and (5.12), we get the following result:

$$A_1(\tau) + \frac{\theta_1}{2}I \overset{\tau\to\infty}{\sim} \frac{R^{-1}}{(AD-BC)}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \theta_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}R$$
$$= R^{-1}\frac{\theta_1}{(AD-BC)}\begin{pmatrix} AD & -AB \\ CD & -CB \end{pmatrix}R \tag{5.22}$$

where

$$A = \left[\tau^{\sigma/2}(ae^\tau + c) - \tau^{-\sigma/2}\left(-\frac{(\sigma+\theta_\infty+\theta_1)}{(\sigma-\theta_\infty-\theta_1)}ae^\tau + c\right)\right]$$
$$B = \left[\tau^{\sigma/2}(be^\tau + d) - \tau^{-\sigma/2}\left(-\frac{(\sigma+\theta_\infty+\theta_1)}{(\sigma-\theta_\infty-\theta_1)}be^\tau + d\right)\right]$$
$$C = \left[\tau^{\sigma/2}\left(\frac{\sigma+\theta_\infty+\theta_1}{\sigma-\theta_\infty-\theta_1}\right)(ae^\tau + c) + \tau^{-\sigma/2}\left(-\frac{(\sigma+\theta_\infty+\theta_1)}{(\sigma-\theta_\infty-\theta_1)}ae^\tau + c\right)\right] \tag{5.23}$$
$$D = \left[\tau^{\sigma/2}\left(\frac{\sigma+\theta_\infty+\theta_1}{\sigma-\theta_\infty-\theta_1}\right)(ae^\tau + d) + \tau^{-\sigma/2}\left(-\frac{(\sigma+\theta_\infty+\theta_1)}{(\sigma-\theta_\infty-\theta_1)}be^\tau + d\right)\right].$$

Here a, b, c, d depend on $\hat{s}$ and can be obtained from relation (5.20a). This gives $A_0(\tau)$, as follows, in the limit $\tau \to \infty$:

$$A_0(\tau) + \frac{\theta_0}{2} I = R^{-1} \begin{bmatrix} -(\theta_\infty + \theta_1)/2 & (\sigma - \theta_\infty - \theta_1)/2 \\ (\sigma + \theta_\infty + \theta_1)/2 & (\theta_\infty + \theta_1)/2 \end{bmatrix} + \frac{\theta_1}{(AD - BC)} \begin{bmatrix} AD & -AB \\ CD & -CB \end{bmatrix} + \begin{bmatrix} \frac{\theta_0 - \theta_1}{2} & \\ & \frac{\theta_0 - \theta_1}{2} \end{bmatrix} R. \tag{5.24}$$

Using (5.22) and (5.24) in (2.24), we get

$$y(\tau) \overset{\tau\to\infty}{\to} \frac{B}{D} \frac{[(\theta_0 - \theta_\infty) AD - (\theta_0 - \theta_\infty - 2\theta_1) BC]}{[2\theta_1 AB - (\sigma - \theta_\infty - \theta_1)(AD - BC)]}. \tag{5.25}$$

Equation (5.25) gives us athe symptotic behaviour of $y(\tau, \zeta)$ in terms of two unknowns σ and $\hat{s}$. But as σ and $\hat{s}$ are part of the monodromy data, and are thus constant with respect to τ, they also appear in the $y(\tau \to 0, \zeta)$ solution given by Jimbo [7]. But Mahoux and Mehta [1] have obtained the solution of (2.10), i.e. S, in the limit $(\tau \to 0)$ and in terms of only one parameter, namely ζ. Then, by using equations (2.13) and (2.14), they also calculated $y(\tau \to 0, \zeta)$. Thus the comparison of the two solutions of y in the $\tau \to 0$ limit will give us σ and $\hat{s}$ in terms of one parameter ζ, which, on substitution in (5.25), gives us the one-parameter solution $y(\tau \to \infty, \zeta)$.

6. Determination of the constant α_1 by using the $y(\tau \to \infty)$ result of P5

The behaviour of $y_r(\tau)$, and $y_i(\tau)$ near $\tau = 0$ for the values of ($\alpha = \frac{1}{32}, \beta = -\frac{1}{32}, \gamma = 0, \delta = -\frac{1}{2}$) is given as follows [1]:

$$y_r(\tau) = 1 + 2\left(\frac{\tau}{2\zeta}\right)^{1/2} + 2\left(\frac{\tau}{2\zeta}\right) + \cdots \tag{6.1}$$

$$y_i(t) = -1 - 2(\zeta\tau/2)^{1/2} - 2(\zeta\tau/2)^2 + \cdots. \tag{6.2}$$

Any one of the above two P5's can be expressed in terms of the other. The calculation of α_1 can be made by comparing the results for $y(\tau \to 0)$ (obtained by Jimbo [7]) with equations (6.1) or (6.2). This determines the unknown constants σ and $\hat{s}$ in (5.24). Relation (2.25) gives four sets of values $(\theta_0, \theta_1, \theta_\infty)$ for $(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{32}, -\frac{1}{32}, 0, -\frac{1}{2}\right)$, namely, $\left(\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}\right)$, $\left(\frac{3}{4}, \frac{1}{4}, 0\right)$, $\left(\frac{1}{4}, \frac{3}{4}, 0\right)$. For the calculation of α_1, let us choose the set $(\theta_0, \theta_1, \theta_\infty) = \left(\frac{1}{4}, \frac{3}{4}, 0\right)$. Then $y(\tau)$ (obtained from Jimbo's result for $A_0(\tau \to 0)$, $A_1(\tau \to 0)$) is

$$1 - 2\left[\frac{(1-\sigma)}{(1+\sigma)} - \frac{\left[\left(\frac{3}{4} - \sigma\right)^2 - \left(\frac{1}{4}\right)^2\right]}{\left[\left(\frac{3}{4} + \sigma\right)^2 - \left(\frac{1}{4}\right)^2\right]}\right] \hat{s}\tau^\sigma + \cdots. \tag{6.3}$$

Comparison of (6.1) and (6.3) gives $\sigma = \frac{1}{2}$ and $\hat{s} = -3/\sqrt{2\zeta}$. $y(\tau)$ in the $\tau \to \infty$ limit can now be calculated (by using 'Mathematica' symbolic programming software) and, for calculating α_1, can be expanded in terms of $(\zeta - 2/\pi)$

$$y = 1 + \left[\left(\frac{2048\beta_1 e^{-\tau}\gamma_1}{3\left(3\sqrt{\pi} - 32\beta_1\right)} + \frac{4096\beta_1\gamma_1^2\left(-21\sqrt{\pi} + 64\beta_1\right)e^{-2\tau}}{15\left(3\sqrt{\pi} - 32\beta_1\right)^2} + O\left(e^{-\tau}\right)^3\right)\tau^{-1} + O\left(\tau^{-1/2}\right)^3\right]$$

$$+ \left[\left(\frac{512(\pi)^{3/2}\beta_1\gamma_1 e^{-\tau}}{\left(-3\sqrt{\pi} + 32\beta_1\right)^2} + \frac{3072\pi^{3/2}\beta_1\gamma_1^2\left(7\sqrt{\pi} + 32\beta_1\right)e^{-2\tau}}{5\left(-3\sqrt{\pi} + 32\beta_1\right)^3} + O\left(e^{-\tau}\right)^3\right)\tau^{-1}\right.$$

$$\left. + \left(\frac{5632(\pi)^{3/2}\beta_1\gamma_1 e^{-\tau}}{\left(-3\sqrt{\pi} + 32\beta_1\right)^2}\right)\tau^{-3/2} + O\left(\tau^{-1/2}\right)^4\right]\left(\zeta - \frac{2}{\pi}\right) + O\left(\zeta - \frac{2}{\pi}\right)^2$$

$$= 1 + H \tag{6.4}$$

where $\gamma_1 = -11\sqrt{\pi}/\sqrt{2}$, and $\beta_1 = \left(\frac{3}{32}\right)\sqrt{\pi}$. But from (2.18) and (6.4), we have

$$(1 + H)\left(R_0\frac{e^{-\tau}\sqrt{\tau}}{\sqrt{2}\alpha_1\left(\zeta - \frac{2}{\pi}\right)} - \frac{e^{-\tau}\sqrt{\tau}}{2\sqrt{2}\alpha_1\left(\zeta - \frac{2}{\pi}\right)} + \sum_{n=1}^{\infty}\left[\alpha_1\left(\zeta - \frac{2}{\pi}\right)\frac{e^{\tau}\sqrt{2}}{\sqrt{\tau}}\right]^{n-1} R_n\left(\frac{\tau}{2}\right)\right)$$

$$= R_0\frac{e^{-\tau}\sqrt{\tau}}{\sqrt{2}\alpha_1(\zeta - \frac{2}{\pi})} + \frac{e^{-\tau}\sqrt{\tau}}{2\sqrt{2}\alpha_1(\zeta - \frac{2}{\pi})} + \sum_{n=1}^{\infty}\left[\alpha_1\left(\zeta - \frac{2}{\pi}\right)\frac{e^{\tau}\sqrt{2}}{\sqrt{\tau}}\right]^{(n-1)} R_n\left(\frac{\tau}{2}\right). \tag{6.5}$$

Substituting the value of H, neglecting all terms containing $e^{-\tau}$ and comparing coefficients of terms containing the same powers of e^{τ}, τ and ζ on both sides gives $\alpha_1 = \sqrt{\pi}/8$. This value of α_1 turns out to be same as that guessed by Mahoux and Mehta [1]. This now completes the determination of $S(\zeta, \tau)$, and hence the spacing distribution of the eigenvalues of random matrix theory in terms of all known constants.

Appendix 1. Proof of separability of ψ

Equation (2.20) reads

$$\frac{\partial\psi(x, \tau)}{\partial\tau} = -\frac{A_1(\tau)}{x - \tau}\psi(x, \tau). \tag{A1.1}$$

Integration of the above equation with respect to τ leads to

$$\psi(x, \tau) = h(x)\, T\exp\left[-\int\frac{A_1(\tau)}{(x - \tau)}\,d\tau\right] \tag{A1.2}$$

$$= h(x)\, T\exp\left[\int\frac{A_1(\tau)}{\tau}\left(1 + \frac{x}{\tau} + \frac{x^2}{\tau^2} + \cdots\right)d\tau\right] \tag{A1.3}$$

$$= h(x)\, T\exp\left[\sum_n^{\infty} x^{n-1}\int\frac{A_1(\tau)}{\tau^n}\,d\tau\right] \tag{A1.4}$$

where 'T' before 'exp' implies τ-ordering, where $h(x)$ is defined by the initial conditions on $\psi(x,\tau)$. Now under the assumption that $\frac{A_1(\tau)}{\tau} \to 0$, the terms with $n \geqslant 1$ in the sum appearing in the exponent of (A1.4) can safely be dropped. This leaves us with the following form of ψ under the limit (3.1):

$$\psi(x,\tau) = h(x)\, T \exp\left[\int \frac{A_1(\tau)}{\tau}\, \mathrm{d}\tau\right]. \tag{A1.5}$$

Hence it is obvious from (A1.5) that in the limit (3.1) $\psi(x,\tau)$ can be separated in terms of the variables x and τ.

Appendix 2. Proof of separability of $\tilde{\psi}$

We have

$$\tilde{\psi}(x,\tau) = \left(\frac{x}{\tau}+\tau\right)^{\Lambda} \mathrm{e}^{-G(x/\tau+\tau)} \psi\left(\frac{x}{\tau}+\tau, \tau, \zeta\right) \tag{A2.1}$$

where the function $\psi(x/\tau+\tau,\tau,\zeta)$ satisfies the following equation:

$$\frac{\partial\psi}{\partial(x/\tau+\tau)} = \left[\frac{A_0(\tau)}{(x/\tau+\tau)} + \frac{A_1(\tau)}{x/\tau} + G\right]\psi \tag{A2.2}$$

$$\frac{\partial\psi}{\partial\tau} = -\frac{A_1(\tau)}{x/\tau}\psi. \tag{A2.3}$$

It is obvious from (A2.3) that the function $\psi(x/\tau+\tau,\tau,\zeta)$ is separable in x and τ. Also, as in the limit $\tau\to\infty$ the function $\left(\frac{x}{\tau}+\tau)\right)^{\Lambda} \to \tau^{\Lambda}$ and the function $\mathrm{e}^{-G(\frac{x}{\tau}+\tau))} \to \mathrm{e}^{-G\tau}$, *it is therefore clear that the function $\tilde{\psi}$ is separable in x and τ in the limit $\tau\to\infty$*. Again

$$\frac{\partial\tilde{\psi}(x,\tau)}{\partial\tau} = \frac{\partial}{\partial\tau}\left[\left(\frac{x}{\tau}+\tau\right)^{\Lambda} \mathrm{e}^{-G(x/\tau+\tau)} \psi\left(\frac{x}{\tau}+\tau,\tau,\zeta\right)\right] \tag{A2.4}$$

$$\begin{aligned} &= \Lambda_1 \log(x/\tau+\tau)\tilde{\psi} + \frac{\Lambda}{(x/\tau+\tau)}\left(-\frac{x}{\tau^2}+1\right)\tilde{\psi} \\ &\quad + \left(\frac{x}{\tau}+\tau\right)^{\Lambda}(-G)\Big(-\frac{x}{\tau^2}+\tau\Big)\mathrm{e}^{-G(x/\tau+\tau)}\psi\left(\frac{x}{\tau}+\tau,\tau,\zeta\right) \\ &\quad + \left(\frac{x}{\tau}+\tau\right)^{\Lambda} \mathrm{e}^{-G(x/\tau+\tau)}\left[\frac{\partial\psi}{\partial(x/\tau+\tau)}\frac{\partial(x/\tau+\tau)}{\partial\tau} + \frac{\partial\psi}{\partial\tau}\right] \end{aligned} \tag{A2.5}$$

where

$$\Lambda_1 = \int_0^1 \tau^{r\Lambda}\frac{\mathrm{d}\Lambda}{\mathrm{d}\tau}\tau^{-r\Lambda}\,\mathrm{d}r.$$

Substituting (A2.2) and (A2.3) in (A2.5), and taking the limit (3.1) leads us to

$$\frac{\partial\tilde{\psi}}{\partial\tau} = \Lambda_1\log\tau + \frac{1}{\tau}\left[-\Lambda + \tau^{\Lambda}\mathrm{e}^{-G\tau}\left(A_0(\tau) - A_1(\tau)\right)\mathrm{e}^{G\tau}\tau^{-\Lambda}\right]\tilde{\psi}. \tag{A2.6}$$

By using (4.6) and (4.8) in (A2.5), we get

$$\frac{\partial\tilde{\psi}}{\partial\tau} = \left[\Lambda_1\log\tau\right]\tilde{\psi}. \tag{A2.7}$$

It therefore follows from (A2.7) that, in the limit (3.1), one can write $\tilde{\psi} = \tilde{\phi}(x)\, T\exp(\int \Lambda_1 \log\tau\, \mathrm{d}\tau) = T\exp(\int \Lambda_1\log\tau\,\mathrm{d}\tau)\tilde{\phi}(x)$ where the matrix $\phi(x)$ is assumed to commute with Λ_1.

Appendix 3. Derivation of equations (5.4) and (5.5) using the Stokes phenomena of W (equation (5.6))

The behaviour of the function $\varphi(x)$ in various sectors can be written as follows (equation (5.6)):

$$\varphi_1(x) = R^{-1}W_1R \tag{A3.1}$$

$$\varphi_2(x) = R^{-1}W_2R. \tag{A3.2}$$

But as a_1 and b_1 are the Stokes multipliers of the function W, the relation between its solution in sectors 1 and 2 is given by

$$W_2(x) = W_1(x)\begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}. \tag{A3.3}$$

This further implies that

$$R^{-1}W_2R = R^{-1}W_1RR^{-1}\begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}R \tag{A3.4}$$

$$= R^{-1}W_1R\begin{pmatrix} 1 & 0 \\ a_1/r & 1 \end{pmatrix}. \tag{A3.5}$$

Now as $\frac{a_1}{r} = a$, this, along with equations (A3.1) and (A3.2), gives the following equality:

$$\varphi_2(x) = \varphi_1(x)\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}. \tag{A3.6}$$

Similarly by using the following relation between the solutions of W in sectors 2 and 3:

$$W_3(x) = W_2(x)\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \tag{A3.7}$$

one can show that

$$\varphi_3(x) = \varphi_2(x)\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}. \tag{A3.8}$$

Appendix 4. Verification of equation (5.11)

The Whittaker function $W_{k,s}$ can be expressed in terms of the hypergeometric function $H(a, c; -x)$ as follows [9]:

$$W_{k,s}(\mathrm{e}^{-\pi\mathrm{i}}x) = \mathrm{e}^{x/2}(-x)^{c/2}H(a, c; -x) \tag{A4.1}$$

where $a = \frac{1}{2} - k + s$ and $c = 2s + 1$. Using equation (A4.1), it can be shown that

$$\frac{\mathrm{d}W_{(1-k)/2,s/2}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{x/2}(-x)^{(s+1)/2}H((k+s)/2, s+1; -x)\right] \tag{A4.2}$$

$$= \mathrm{e}^{x/2}(-x)^{(s+1)/2}\frac{\mathrm{d}}{\mathrm{d}x}H((k+s)/2, s+1; -x) + \frac{W}{2} + \frac{(s+1)W}{2x}. \tag{A4.3}$$

But

$$\frac{d}{dx}H(a,c;-x) = -\frac{\Gamma(a+1)}{\Gamma(a)}H(a+1,c+1;-x) \tag{A4.4}$$

and

$$xH(a+1,c+1;-x) = (c-a-1)H(a+1,c;-x) + H(a,c;-x). \tag{A4.5}$$

Using these relations in (A4.3), We get

$$\frac{dW_{(1-k)/2,s/2}}{dx} = \frac{(s+k)}{2x}W_{(1-k)/2,s/2} - \frac{(s+k)}{2x}\frac{(s-k)}{2}W_{-(1+k)/2,s/2} + \frac{W_{(1-k)/2,s/2}}{2} + \frac{(s+1)W_{(1-k)/2,s/2}}{2x}. \tag{A4.6}$$

Now if we denote $\varphi_1(x)$ as

$$\varphi_1(x) = \begin{pmatrix} a1 & a2 \\ a3 & a4 \end{pmatrix} \tag{A4.7}$$

then

$$\frac{da1}{dx} = \frac{d}{dx}\left[e^{\pi i(1-\theta_\infty-\theta_1)/2}W_{(1-\theta_\infty-\theta_1)/2,\sigma/2}(e^{-\pi i}x)\right], \tag{A4.8}$$

$$= \frac{(\sigma-\theta_\infty-\theta_1)}{2x}a3 - \frac{(\sigma+\theta_\infty+\theta_1)}{2x}a1 + \frac{a1}{2} + \frac{(\sigma+1)}{2x}a1. \tag{A4.9}$$

But this is also the behaviour of the differential equation for $a1$, given by equations (5.6) and (5.11). In the same way one can obtain differential equations for $a2, a3, a4$ and show that that they match (5.11).

Appendix 5. Monodromy data for $\tilde{\psi}$

Let us consider the function

$$\tilde{\psi}(x,\tau,\zeta) = \left(\frac{x}{\tau}+\tau\right)^{\Lambda} e^{-G(x/\tau+\tau)}\psi\left(\frac{x}{\tau}+\tau,\tau,\zeta\right)$$
$$= \tilde{x}^{\Lambda}e^{-G\tilde{x}}\psi(\tilde{x},\tau,\zeta) \tag{A5.1}$$

where $\tilde{x} = x/\tau + \tau$.

The local behaviour of $\tilde{\psi}(x,\tau,\zeta)$ around $x \to 0$ can be written as follows:

$$\tilde{\psi}(x\to 0,\tau,\zeta) = \tau^{\Lambda}e^{-G\tau}\psi(\tilde{x}\to\tau,\tau,\zeta). \tag{A5.2}$$

Now by using (2.32), the local behaviour of the function $\psi(\tilde{x},\tau,\zeta)$ around the singularity $\tilde{x} = \tau$ can be written as

$$\psi(\tilde{x},\tau,\zeta) = G_1(\tau)[1+O(\tilde{x}-\tau)](\tilde{x}-\tau)^{T_1}C_1 \tag{A5.3}$$

$$= G_1(\tau)\left[1+O\left(\frac{x}{\tau}\right)\right]\left(\frac{x}{\tau}\right)^{T_1}C_1 \tag{A5.4}$$

$$= G_1(\tau)\tau^{-T_1}x^{T_1}C_1. \tag{A5.5}$$

Thus substitution of (A5.5) in (A5.2) gives us

$$\tilde{\psi}(x \to 0, \tau, \zeta) = \tau^{\Lambda} e^{-G\tau} G_1(\tau) \tau^{-T_1} x^{T_1} C_1. \tag{A5.6}$$

Note that in both (A5.5) and (A5.6), the eigenvalue matrix T_1 and connection matrix C_1 is the same. Also, the behaviour of $\tilde{\psi}$, when taken around the singularity $x = 0$ in the complex x-plane, can be given as

$$\tilde{\psi}(x e^{2\pi i}, \tau, \zeta) = \tau^{\Lambda} e^{-G\tau} G_1(\tau) \tau^{-T_1} x^{T_1} e^{2\pi i T_1} C_1 \tag{A5.7}$$

$$= \left[\tau^{\Lambda} e^{-G\tau} G_1(\tau) \tau^{-T_1} x^{T_1} C_1\right] C_1^{-1} e^{2\pi i T_1} C_1 \tag{A5.8}$$

$$= \tilde{\psi}(x, \tau, \zeta) M_1. \tag{A5.9}$$

Here M_1 is monodromy matrix of $\psi(x, \tau, \zeta)$ around the singularity $x = \tau$, described by (2.36). Hence, the function $\tilde{\psi}(x, \tau, \zeta)$ has the same monodromy matrix around the singularity $x = 0$ as that of $\psi(x, \tau, \zeta)$ around the singularity $x = \tau$.

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